Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel
Level 3 GCE

Centre Number	Candidate Number

Specimen Paper

(Time: 1 hour 30 minutes) | Paper Reference **9FM0/02**

Further Mathematics

Advanced

Paper 2: Core Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

1. (a) Find

$$\int \frac{1}{x^2+6x+25}\,dx$$

(3)

(b) Hence find the exact value of

$$\int_{-3}^{1}\left(1-\frac{25}{x^2+6x+25}\right)dx$$

giving the answer in simplest form.

(3)

A student claims that the magnitude of the answer to part (b) gives the total area bounded by the curve $y=1-\dfrac{25}{x^2+6x+25}$ and the x-axis between the line $x=-3$ and the line $x=1$

(c) State, with a reason, whether or not the student is correct.

(1)

(Total for Question 1 is 7 marks)

(a) notice we are asked to integrate a fractional expression ∴ checking if can use the following:

Fractional expressions

4a. Can I split the numerator?
Is there a single term in the denominator?
4b. Can I do partial fractions?
Does the denominator factorise?
4c. Can I do algebraic division?
Is the fraction improper?

explained more in detail on pg. 3

... but:

· can't separate numerator as 1 in the numerator rather than a fractional expression

· can't do partial fractions as checking discriminant for quadratic:

$(6)^2-4(1)(25) = 36-100$
$= -64 < 0$
$\Rightarrow$ cannot factorise

· also can't do polynomial division as 1 in the numerator

↳ this means our only option is to complete the square in the denominator and look to use one of the integrals in the formula booklet – step 7

$$\int \frac{1}{(x+3)^2-9+25}\,dx = \int \frac{1}{(x+3)^2+16}\,dx$$

↳ notice this is in the format $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$; now two ways to proceed:-

WAY 1: formula booklet and equation

where $x \to x+3$ and $a^2 = 16$

$\Rightarrow a = 4$

$\therefore$ subbing into integral

$$\boxed{\frac{1}{4}\arctan\left(\frac{x+3}{4}\right) + c}$$

WAY 2: by substitution

let $u = x+3$

$\frac{du}{dx} = 1 \Rightarrow du = dx$

subbing into integral

$$\int \frac{1}{u^2+16}\,du$$

subbing into formula book integration

where $a^2 = 16$

$a = 4$

$$= \frac{1}{4}\arctan\left(\frac{u}{4}\right) + c$$

subbing $u = x+3$ back in

$$\boxed{= \frac{1}{4}\arctan\left(\frac{x+3}{4}\right) + c}$$

(b) $\displaystyle\int_{-3}^{1}\left(1 - \frac{25}{x^2+6x+25}\right)dx$

notice this is 25 times part (a)

$$\left[x - \frac{25}{4}\arctan\left(\frac{x+3}{4}\right)\right]_{-3}^{1}$$

evaluate at limits

$$\left\{\left[(1) - \frac{25}{4}\arctan\left(\frac{1+3}{4}\right)\right] - \left[-3 - \frac{25}{4}\arctan\left(\frac{-3+3}{4}\right)\right]\right\}$$

$$= 1 - \frac{25}{4}\tan^{-1}(1) + 3 + \frac{25}{4}\tan^{-1}(0)$$

$$= 1 - \frac{25}{4}(\pi/4) + 3 + 0$$

$$\boxed{= 4 - \frac{25\pi}{16}}$$

Reminders:

Students find fractions tough as fractions can be so many types.

Check first (and throughout the question) if you can simplify by:

- using basic indices rules to simplify and expand brackets
  - $x^a \times x^b = x^{a+b}$
  - $\frac{x^a}{x^b} = x^{a-b}$
  - $\frac{3}{5x}$ means $\frac{3}{5}x^{-1}$.
  - $(\sqrt[b]{x})^a$ or $\sqrt[b]{x^a} = x^{\frac{a}{b}}$
- Factorising and maybe cancel first
- Is there a single term in denominator?

  split fractions using $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ or $(a+b)c^{-1}$

Then ask yourself:

1. **Is it an easy power type?** $\int x^n dx = \frac{x^{n+1}}{n+1}$
2. **Is it ln (natural logarithm)?** Form $\int \frac{f'(x)}{f(x)}dx$

   To recognize these, the power in the denominator is (almost always) 1. When you bring the denominator up to the numerator using negative power indices rule you get a power of -1. By adding one to the power and dividing it, you'll end up dividing by zero which you can't do

   $$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + C$$

   Method: copy $\ln$(denominator). Remember ignore then differentiate to check you get what is inside the integral - correct with numbers only, not variables and only correct by multiplying or dividing. We can ignore the pink part since the derivative 'pops' out when we differentiate and we know when we differentiate our answer it must be what is inside the integral).
3. **Is it bring up and harder power type?** Bring the power up and becomes the form $\int f'(x)f(x)^n\,dx = \frac{f(x)^{n+1}}{n+1} + C$

   Recognisable by a power in the denominator other than

   $\int \frac{4x}{(2x^2-1)^3} = \int 4x(2x^2-10)^{-3}\,dx$ etc
4. **Is it Partial fractions!** Recognisable by products in the denominator.

   Form 1 $\frac{\dots}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$

   Form 2 $\frac{\dots}{(dx+e)(fx+g)^2} = \frac{A}{dx+e} + \frac{B}{fx+g} + \frac{C}{(fx+g)^2}$

   (only advanced courses have this form)

   Form 3 $\frac{\dots}{(dx+e)(fx^2+g)} = \frac{A}{dx+e} + \frac{Bx+C}{fx^2+g}$
5. **Is it divide first?** Recognisable by **two or more terms** in the denominator and also where we have the matching highest powers in both numerator and denominator or a higher power in the numerator
6. **Rewriting/adapting fraction in a clever way (split up the numerator to get two fractions)**
7. **Is it inverse trig? (may need to complete the square first)** Either use the inverse trig results below or use a trig substitution

$$\int \frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int -\frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int \frac{1}{a^2+(bx)^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + C$$

(c) 'magnitude of answer' suggests the graph potentially going under the x-axis (-ve area ∴ need to split the limits)

let $f(x) = 1 - \frac{25}{x^2+6x+25}$

and evaluate at $x = -3$

$$f(-3) = 1 - \frac{25}{(-3)^2 + 6(-3) + 25}$$

$$= 1 - \frac{25}{9 - 18 + 25}$$

$$= 1 - \frac{25}{16}$$

getting common denominator

$$= \frac{1(16) - 25}{16} = -\frac{9}{16}$$

$$f(1) = 1 - \frac{25}{(1)^2 + 6(1) + 25}$$

$$= 1 - \frac{25}{32}$$

$$= \frac{32}{32} - \frac{25}{32} = \frac{7}{32}$$

$\boxed{\text{since } f(x) \text{ changes sign across the interval} \Rightarrow \text{graph crosses the } x\text{-axis} \therefore \text{ part of area is below the } x\text{-axis (-ve area)} \therefore \text{ statement incorrect}}$

2. A company operating a coal mine is concerned about the mine running out of coal. It is estimated that 2.5 million tonnes of coal are left in the mine. The company wishes to mine all of this coal in 20 years.

In order to mine the coal in a regulated manner, the company models the amount of coal to be mined in the coming years by the formula

$$M_r = \frac{10}{r^2 + 8r + 15}$$

where M_r is the amount of coal, in millions of tonnes, mined in year r, with the first year being year 1

(a) Show that, according to the model, the total amount of coal, in millions of tonnes, mined in the first n years is given by

$$T_n = \frac{9n^2 + 41n}{k(n+4)(n+5)}$$

where k is a constant to be determined.

(6)

(b) Explain why, according to this model, the mine will never run out of coal.

(2)

The company decides to mine an extra fixed amount each year so that all the coal will be mined in exactly 20 years.

(c) Refine the formula for M_r so that 2.5 million tonnes of coal will be exhausted in exactly 20 years of mining.

(2)

(Total for Question 2 is 10 marks)

(a) METHOD 1: method of differences (summations)

'total amount of coal' suggests need a summation of M_r from $r=1$ to $r=n$

∴ using Sigma notation

$$\sum_{r=1}^{n} \frac{10}{r^2+8r+15}$$

straight away notice fraction ∴ implies need for partial fractions (FACTORISE)

$$\frac{10}{(r+3)(r+5)} = \frac{A}{r+3} + \frac{B}{r+5}$$

$$\Rightarrow 10 = A(r+5) + B(r+3)$$

WAY 1: compare coefficients

... r:

$0 = A + B$ -①

... 'constants':

$10 = 5A + 3B$ -②

solve simultaneously - equtn solver calc

or by elimination

② − 3×①

$$\begin{array}{r} 5A + 3B = 10 \\ -\ 3A + 3B = 0 \\ \hline 2A = 10 \end{array}$$

$\div 2 \quad \div 2$

WAY 2: by substitution - making each bracket $=0$

let $r=-5$,

$10 = -2B$

$\div -2 \quad \div -2$

$\Rightarrow B = -5$

let $r=-3$,

$10 = 2A$

$\div 2 \quad \div 2$

$\Rightarrow A=5$

subbing into ① for B:

$0 = 5 + B$

$\Rightarrow B = -5$

$A = 5$

$$\therefore \frac{10}{(r+3)(r+5)} = \underbrace{\frac{5}{r+3} - \frac{5}{r+5}}$$

notice this is in the form $f(r) - f(r+2)$

$\therefore$ hinting at need to exploit methods of differences techniques

finding common factor of '5' and using partial fractions

$$5\sum_{r=1}^{n} \frac{1}{r+3} - \frac{1}{r+5}$$

...two ways to evaluate this:

WAY 1: numerically

$u_1 : \frac{1}{1+3} - \frac{1}{1+5}$

$= \frac{1}{4} - \cancel{\frac{1}{6}}$

$u_2 : \frac{1}{2+3} - \frac{1}{2+5}$

$= \frac{1}{5} - \cancel{\frac{1}{7}}$

$u_3 : \frac{1}{3+3} - \frac{1}{3+5}$

$= \cancel{\frac{1}{6}} - \cancel{\frac{1}{8}}$

$\vdots$

$u_{n-2} : \frac{1}{n-2+3} - \frac{1}{n-2+5}$

$= \frac{1}{n+1} - \cancel{\frac{1}{n+3}}$

$u_{n-1} : \frac{1}{n-1+3} - \frac{1}{n-1+5}$

$= \cancel{\frac{1}{n+2}} - \frac{1}{n+4}$

$u_n : \cancel{\frac{1}{n+3}} - \frac{1}{n+5}$

cancelling terms

...left with:

$$5\left(\frac{1}{4} + \frac{1}{5} - \frac{1}{n+4} - \frac{1}{n+5}\right)$$

getting common denominator:

$$5\left(\frac{5(n+4)(n+5) + 4(n+4)(n+5) - 20(n+5) - 20(n+4)}{20(n+4)(n+5)}\right)$$

expand numerator:

$$5\left(\frac{5(n^2+9n+20)+4(n^2+9n+20)-20n-100-20n-80}{20(n+4)(n+5)}\right)$$

$$=5\left(\frac{5n^2+45n+100+4n^2+45n+80-20n-100-20n-80}{20(n+4)(n+5)}\right)$$

$$=\cancel{5}\left(\frac{9n^2+41n}{\cancel{20}_{4}(n+4)(n+5)}\right)$$

$$=\frac{9n^2+41n}{4(n+4)(n+5)}$$

$$\Rightarrow \boxed{k=4}$$

WAY 2: mechanically

let $f(r)=\frac{1}{r+3}$, $f(r+2)=\frac{1}{r+5}$

$$\sum_{r=1}^{n} f(r)-f(r+2)$$

evaluate above for $r=1,2,3,\ldots,n-2,n-1,n$

$u_1:\ f(1)-f(1+2)$
$\quad =f(1)-\cancel{f(3)}$

$u_2:\ f(2)-f(2+2)$
$\quad =f(2)-\cancel{f(4)}$

$u_3:\ \cancel{f(3)}-f(3+2)$
$\quad =f(3)-\cancel{f(5)}$

$\vdots$

$u_{n-2}:\ f(n-2)-f(n-2+2)$
$\quad =f(n-2)-\cancel{f(n)}$

$u_{n-1}:\ f(n-1)-f(n-1+2)$
$\quad =f(n-1)-f(n+1)$

$u_n:\ f(\cancel{n})-f(n+2)$

$\therefore f(1)+f(2)-f(n+1)-f(n+2)$

subbing into previously defined function

$$\frac{1}{4}+\frac{1}{5}-\frac{1}{n+4}-\frac{1}{n+5}$$

and manipulate as above to get

$$\boxed{k=4}$$

Alternative method to part (a) - use of induction

↳ finding the value of 'k' using n=1 and the following inductive hypothesis

step 1: base case

prove true for n=1

... M_r:

$$= \frac{10}{(1)^2 + 8(1) + 15}$$

$$= \frac{10}{1+8+15} = \frac{10}{24}$$

... T_r:

$$\frac{9(1)^2 + 41(1)}{k(1+4)(1+5)}$$

$$= \frac{9+41}{k(5)(6)} = \frac{50}{30k}$$

equating these

$$\frac{10}{24} = \frac{50}{30k}$$

cross multiply

$$300k = 1200$$

$\div 300$ $\div 300$

$$k = \frac{12\cancel{00}}{3\cancel{00}} = 4$$

step 2: assumption step

assume true for $n=p$ (AVOID 'k' as can get confused with k=4)

$$\sum_{r=1}^{k} M_r = \frac{9p^2+41p}{4(p+4)(p+5)}$$

step 3: induction step

prove true for $n=k+1$

$$\sum_{r=1}^{k+1} M_r = \sum_{r=1}^{k} M_r + \sum_{r=1}^{1} M_r$$

$$= \frac{9p^2+41p}{\text{“}4\text{”}(p+4)(p+5)} + \frac{10}{(p+1)^2+8(p+1)+15}$$

expand denominator of second fraction

$$= \frac{9p^2+41p}{4(p+4)(p+5)} + \frac{10}{p^2+2p+1+8p+8+15}$$

collect like terms

$$= \frac{9p^2+41p}{4(p+4)(p+5)} + \frac{10}{p^2+10p+24}$$

AIM:

$$\frac{9(p+1)^2+41(p+1)}{4((p+1)4)((p+1)+5)}$$

$$= \frac{9(p^2+2p+1)+41p+41}{4(p+5)(p+6)}$$

$$= \frac{9p^2+18p+9+41p+41}{4(p+5)(p+6)}$$

$$= \frac{9p^2+59p+50}{4(p+5)(p+6)}$$

$$= \frac{(p+1)(9p+50)}{4(p+5)(p+6)}$$

factorise second fraction's denominator:

$$= \frac{9p^2+41p}{4(p+4)(p+5)} + \frac{10}{(p+6)(p+4)}$$

...getting common denominator:

$$= \frac{(9p^2+41p)(p+6) + 10(4(p+5))}{4(p+4)(p+5)(p+6)}$$

expand top numerator

$$\frac{9p^3+41p^2+54p^2+246p+40p+200}{4(p+4)(p+5)(p+6)}$$

collect like terms

$$\frac{9p^3+95p^2+286p+200}{4(p+4)(p+5)(p+6)}$$

factorise numerator - calc equtn solver

$$\frac{(p+1)\cancel{(p+4)}(9p+50)}{4\cancel{(p+4)}(p+5)(p+6)}$$

$$= \frac{(p+1)(9p+50)}{4(p+5)(p+6)} \quad = \text{AIM } (\checkmark)$$

$\therefore$ true for $n=k+1$

step 4: conclusion step

$\boxed{\text{since true for } n=1, \text{ if true for } n=k \text{ and true for } n=k+1, \text{ then true for all } n \in \mathbb{Z}^+}$

(b) 'never runs out of coal' suggests the need to see what happens to T_n as $t \to \infty$ → given:

$$T_n = \frac{9n^2+41n}{4(n+4)(n+5)}$$ and L'hospital rule ($\div n^2$)

$$\lim_{t\to\infty} \frac{\frac{9n^2+41n}{n^2}}{\frac{4(n+4)(n+5)}{n^2}} = \frac{9+\frac{41}{n}}{\frac{4(n+4)(n+5)}{n^2}}$$

as $t \to \infty$, $T_n \to 9/4 = 2.25$ mlns of tonnes

↳ since $2.25 < 2.5$, mine will never run out of coal

(c) according to current model, when in the first 20 years...

$$T_{20} = \frac{9(20)^2+41(20)}{4(20+4)(20+5)} = \frac{221}{120}$$

to work out additional coal mined - need

$$2.5 - \frac{221}{120} = \frac{79}{120} \text{ mln tonnes over 20 yrs}$$

$$\frac{79}{120} \div 20 = \frac{79}{120(20)}$$

$$= \frac{79}{2400} \text{ tonnes}$$

$$\boxed{\Rightarrow M_r = \frac{79}{2400} + \frac{10}{r^2+8r+15}}$$

3.

$$\mathbf{P} = \begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}$$

The matrix **P** represents a linear transformation, T, of the plane.

(*a*) Describe the invariant points of the transformation T.

(3)

(*b*) Describe the invariant lines of the transformation T.

(6)

(Total for Question 3 is 9 marks)

(a) an invariant point is a point (call it $\begin{pmatrix} x \\ y \end{pmatrix}$) which under the transformation P would be mapped to exactly the same point $\begin{pmatrix} x \\ y \end{pmatrix}$

formulating this as an equation ($Mx = y$)

$$\begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

matrix multiplication "rows into columns" on LHS – let product matrix be $\begin{pmatrix} A_{(1,1)} \\ A_{(2,1)} \end{pmatrix}$

...for $A_{(1,1)}$:

$3(x) + 3(y)$

$\Rightarrow 3x + 3y$

...for $A_{(2,1)}$:

$4(x) + 7(y)$

$= 4x + 7y$

equating to RHS

$$\begin{pmatrix} 3x + 3y \\ 4x + 7y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

formulating 2 sets of linear equations:

$3x + 3y = x$

collect like terms – ①

$\underline{2x = -3y}$

$4x + 7y = y$ – ②

$4x = -6y$

$\div 2$ $\quad \div 2$

$\underline{2x = -3y}$

making 'y' the subject of both:

$y = -\frac{2}{3}x$

$\boxed{\therefore \text{all the points of the line } y = -\tfrac{2}{3}x \text{ are invariant}}$

(b) METHOD 1: an invariant line is a line of points (call it $y = mx + c$ $\therefore$ as a vector

$\begin{pmatrix} x \\ mx+c \end{pmatrix}$) which under the transformation P would be mapped to different points on the SAME straight line

↳ formulating this as an equation (using $Mx = y$)

$$\begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$$

matrix multiplication "rows into columns" - let product matrix be $\begin{pmatrix} A(1,1) \\ A(2,1) \end{pmatrix}$

...for $A(1,1)$:

$3(x) + 3(mx+c)$

expand brackets

$3x + 3mx + 3c$

..for $A(2,1)$:

$4(x) + 7(mx+c)$

expand brackets

$4x + 7mx + 7c$

and equating to RHS:

$$\begin{pmatrix} 3x+3mx+3c \\ 4x+7mx+7c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$$

$3x + 3mx + 3c = x'$ - ①

$4x + 7mx + 7c = mx' + c$ - ②

subbing ① into ②

$4x + 7mx + 7c = m(3x + 3mx + 3c) + c$

expand brackets

$4x + 7mx + 7c = 3mx + 3m^2x + 3mc + c$

collect x's and c's on either side:

$x(4 + 4m - 3m^2)x + c(6 - 3m)c = 0$

$\Rightarrow x(4 + 4m - 3m^2)x + 3(2-m)c = 0$

making each bracket equal 0

...x:

$4 + 4m - 3m^2 = 0$

calc equtn solver to factorise:

$(m-2)(3m+2) = 0$

making each bracket equal 0

$m - 2 = 0 \quad 3m + 2 = 0$

$\Rightarrow m = 2 \quad \Rightarrow m = -2/3$

...when $m = 2$, $3(2-2)$ so 'c' can take any value $\Rightarrow$ $\boxed{y = 2x + c}$

...when $m = -2/3$, $3(2 - (-2/3)) \neq 0$

$\Rightarrow c$ has to be equal to 0 $\therefore \boxed{y = -\tfrac{2}{3}x}$

METHOD 2: transformation of points

an invariant line is a line of points - call them $\begin{pmatrix} x \\ y \end{pmatrix}$, each of which under the transformation are mapped to another point $\begin{pmatrix} x' \\ y' \end{pmatrix}$ on the same line : $y = mx + c$

formulating this as an equation:

$$\begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

using matrix multiplication "rows into columns" on LHS and equate to RHS

$$\begin{pmatrix} 3x + 3y \\ 4x + 7y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

into linear equations

$$3x + 3y = x'$$
$$4x + 7y = y'$$

now subbing in $y = mx + c$ in both so lie on same straight line

$$3x + 3(mx + c) = x'$$

expand brackets

$$- 3x + 3mx + 3c = x'$$
$$4x + 7(mx + c) = y'$$
$$\Rightarrow 4x + 7mx + 7c = y'$$

factorise x's and c's

$$3x(1+m) + 3c = x' \quad x(4+7m) + 7c = y'$$

and sub into transformed points: $y' = mx' + c$

$$x(4+7m) + 7c = 3x(m+m^2) + 3m$$

...compare coefficients:

...x:

$$4 + 7m = 3(m + m^2)$$
$$3m^2 - 4m - 4 = 0$$

factorise

$$(m-2)(3m+2) = 0$$

$\Rightarrow m = 2$, looking at c component

$$6 = 3(2)$$

$= 6 \Rightarrow$ 'c' takes any value $\therefore \boxed{y = 2x + c}$

$$3m + 2 = 0$$

$\Rightarrow m = -\tfrac{2}{3}$, $c = 0$ $\therefore \boxed{y = -\tfrac{2}{3}x}$

...constants:

$$\div 3 \quad 6 = 3m \quad \div 3$$
$$m = 2$$

4. (a) Using the identity $zz^* = |z|^2$, or otherwise, show that if w is any root of unity then

$$|w-2|^2 = 5 - 2(w + w^*)$$

(3)

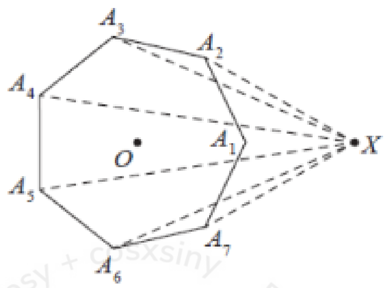

Figure 1

Figure 1 shows a regular heptagon $A_1\,A_2\,A_3\,A_4\,A_5\,A_6\,A_7$ whose vertices all lie on the unit circle with centre at the origin O and A_1 at $(1, 0)$. The point X lies in the same plane as the heptagon and has coordinates $(2, 0)$.

Using the result given in part (a),

(b) find $\sum_{i=1}^{7}(XA_i)^2$

(4)

(Total for Question 4 is 7 marks)

(a) METHOD 1: subbing into identity - $|w-2|^2 = 5-2(w+w^*)$

notice LHS of the equation matches the RHS of the given identity (the $|z|^2$) ∴ subbing $z = w-2$ into the identity

$|w-2|^2 = (w-2)(w-2)^*$

using the distributivity of the conjugates

$|w-2|^* = (w-2)(w^*-2)$

expanding RHS

$\Rightarrow ww^* - 2w - 2w^* + 4$

using $ww^* = |w|^2$ from given identity

$|w|^2 + 4 - 2(w+w^*)$

using fact that w is any root of unity ($w^n = 1$) ∴ modulus of this is always 1

$1 + 4 - 2(w+w^*)$

$\boxed{= 5 - 2(w+w^*) = RHS}$

METHOD 2: using $a+bi$ of complex number form and finding the modulus

let $w = x+iy$ - subbing this into LHS of the equation

$|x+i \;\; -2|^2$ - collecting real and imaginary parts

$|(x-2)^2+iy|$ and evaluating this 'modulus square' of complex numbers:

$$\left(\sqrt{(x-2)^2+y^2}\right)^2=(x-2)^2+y^2$$

using Binomial expansion on bracket

Pascal's triangle

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

$$1x^2+2x(-2)+1(-2)^2+y^2=x^2-4x+4+y^2$$

Splitting the quadratic in x using fact that $z^2-(\alpha+\beta)z+\alpha\beta$ where $\alpha+\beta=2x_\alpha$ (if $\alpha=x+iy$, $\beta=x-iy$)

$$=\underbrace{x^2+y^2}_{\text{recognising identity}}+4-2\underbrace{(x+iy+x-iy)}_{\text{rewriting using substitution}}$$

$$=1+4-2(u+u^*)$$

$$=\boxed{5-2(u+u^*)}=RHS$$

(b) we know that only the results of $z^7=1$ will give us a regular heptagon as shown in Fig 1, with origin at the centre and one vertex at $(1,0)$

$\Rightarrow$ see A_i (where $1\le i\le 7$) are the 7th roots of unity, properties of which are included in answer to part (a)

$\therefore$ rewriting

$$\sum_{i=1}^{7}(XA_i)^2=\sum_{i=1}^{7}(u_i-2)^2$$

and using RHS of proved (a)

$$\sum_{i=1}^{7}\left(5-2(u_i+u_i^*)\right)=\sum_{i=1}^{7}5-2\sum_{i=1}^{7}(u_i+u_i^*)$$

...from this:

$$\sum_{i=1}^{7}5=7\times5=35$$

$$-2\sum_{i=1}^{7}(u_i+u_i^*)$$

↳ interpreting this as summation of complex roots of unity and their complex conjugates $\therefore$ can use fact that sum of roots and conjugate roots of unity $=0$

$$\boxed{\therefore\sum_{i=1}^{7}(XA_i)^2=35}$$

5.

$$y = \arctan(\sinh(x))$$

(a) Show that $\dfrac{d^3y}{dx^3} = \dfrac{dy}{dx} - 2\left(\dfrac{dy}{dx}\right)^3$

(7)

(b) Hence find $\dfrac{d^5y}{dx^5}$ in terms of $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ and $\left(\dfrac{d^3y}{dx^3}\right)^3$

(4)

(c) Find the Maclaurin series for y, in ascending powers of x, up to and including the term in x^5

(3)

(Total for Question 5 is 14 marks)

(a) $y = \arctan(\sinh(x))$

→ to get $\dfrac{d^3y}{dx^3}$ need to get $\dfrac{dy}{dx}$ first

2 ways to differentiate above:

METHOD 1: formula book differentiation: $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$ and chain rule

let $u = \sinh x$

$\dfrac{du}{dx} = \cosh x$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1+u^2} \times \cosh x$

subbing $u = \sinh x$

$= \dfrac{1}{1+\sinh^2 x} \times \cosh x$

$= \dfrac{\cosh x}{1+\sinh^2 x}$

using $\cosh^2 x - \sinh^2 x = 1$ identity REARRANGED

$\Rightarrow \cosh^2 x = 1 + \sinh^2 x$

$= \dfrac{\cosh x}{\cosh^2 x} = \dfrac{1}{\cosh x} = \operatorname{sech} x$

METHOD 2: using implicit differentiation

taking tan of both sides

$\tan y = \tan(\arctan(\sinh x))$

$\Rightarrow \tan y = \sinh x$

using implicit differentiation

($\dfrac{d}{dx}\tan x = \sec^2 x$ and $\dfrac{d}{dx}\sinh x = \cosh x$)

$\sec^2 y \dfrac{dy}{dx} = \cosh x$

$\div \sec^2 y \qquad \div \sec^2 y$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\cosh x}{\sec^2 y}$

using $\sec^2 y = 1 + \tan^2 y$

$= \dfrac{\cosh x}{1+\tan^2 y}$

and getting $\tan^2 y$ from

$\tan y = \sinh x$

$\Rightarrow \tan^2 y = \sinh^2 x$

subbing in: $\dfrac{\cosh x}{1+\sinh^2 x}$

notice denominator can be rewritten

$\cosh^2 x - \sinh^2 x = 1$

$\Rightarrow \cosh^2 x = 1 + \sinh^2 x$

$\Rightarrow \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x} = \operatorname{sech} x$

$\Rightarrow \frac{dy}{dx} = \operatorname{sech} x = (\cosh x)^{-1}$

now two ways to find second derivative:

WAY 1: memorised from normal trig

know from normal trig that $\frac{d}{dx}(\sec x) = \sec x \tan x$

and using the hyperbolic version of this

$\Rightarrow \frac{d^2y}{dx^2} = -\operatorname{sech} x \tanh x$

WAY 2: using chain rule on $\frac{dy}{dx}$

$\frac{dy}{dx} = (\cosh x)^{-1}$

$\frac{d^2y}{dx^2} = (\cosh x)^{-2}$ × inner derivative i.e. $\sinh x$

$= -\frac{1}{\cosh^2 x} \times \sinh x$

$= -\frac{1}{\cosh x} \times \frac{\sinh x}{\cosh x}$

$= -\operatorname{sech} x \tanh x$

$\frac{d^2y}{dx^2} = -\operatorname{sech} x \tanh x = -\frac{dy}{dx} \tanh x$

finally third derivative: **WAY 1:**

$\frac{d^3y}{dx^3} = \frac{d}{dx}(-\operatorname{sech} x \tanh x)$

↳ using product rule:

$u = -\operatorname{sech} x \qquad v = \tanh x$

$u' = \operatorname{sech} x \tanh x \qquad v' = \operatorname{sech}^2 x$

$= \operatorname{sech} x \tanh x \tanh x - \operatorname{sech} x \operatorname{sech}^2 x$

$= \operatorname{sech} x \tanh^2 x - \operatorname{sech}^3 x$

but need to manipulate above in terms of $\frac{dy}{dx} = \operatorname{sech} x$ and $\left(\frac{dy}{dx}\right)^3 = (\operatorname{sech} x)^3$ - hence need to use identity $\operatorname{sech}^2 x = 1 - \tanh^2 x$

$\Rightarrow \tanh^2 x = 1 - \operatorname{sech}^2 x$

$(1 - \operatorname{sech}^2 x) \operatorname{sech} x - \operatorname{sech}^3 x$

expand

$\operatorname{sech} x - \operatorname{sech}^3 x - \operatorname{sech}^3 x$

$\operatorname{sech} x - 2\operatorname{sech}^3 x$

$\Rightarrow \boxed{\frac{d^3y}{dx^3} = \frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^3}$

WAY 2: differentiating derivatives

now third derivative: using product rule

$u = -\tanh x \qquad v = \frac{dy}{dx}$

$u' = -\operatorname{sech}^2 x \qquad v' = \frac{d^2y}{dx^2}$

$\therefore \frac{d^3y}{dx^3} = -\tanh\left(\frac{d^2y}{dx^2}\right) - \operatorname{sech}^2 x \frac{dy}{dx}$

subbing in $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$

$$= -\tanh x\left(-\tanh x\frac{dy}{dx}\right) - \operatorname{sech}^2 x(\operatorname{sech} x)$$

$$= \tanh^2 x \operatorname{sech} x - \operatorname{sech}^3 x$$

and manipulate as shown in WAY 1 to finally get $\boxed{\frac{d^3y}{dx^3} = \frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^3}$

(b) using differentiation of derivatives - from part (a)

$$\frac{d^3y}{dx^3} = \frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^3$$

differentiating this using $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ and $\frac{d}{dx}\left(\frac{dy}{dx}\right)^3$ by chain rule

$$\frac{d^4y}{dx^4} = \frac{d^2y}{dx^2} - 2(3)\left(\frac{dy}{dx}\right)^2 \times \frac{d^2y}{dx^2}$$

$$= \frac{d^2y}{dx^2} - 6\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2}$$

now have to differentiate this - using $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ and product rule :

$$u = \left(\frac{dy}{dx}\right)^2 \qquad v = \frac{d^2y}{dx^2}$$

$$u' = 2\left(\frac{dy}{dx}\right) \times \frac{d^2y}{dx^2} \qquad v' = \frac{d^3y}{dx^3}$$

$$\frac{d^5y}{dx^5} = \frac{d^3y}{dx^3} - 6\left(2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \frac{d^3y}{dx^3}\right)$$

expand and simplify

$$\boxed{= \frac{d^3y}{dx^3} - 12\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)^2 - 6\left(\frac{dy}{dx}\right)^2 \frac{d^3y}{dx^3}}$$

(c) know the general Maclaurin series as an infinitely long polynomial where all the coefficients of powers of x are determined by $f(x)$ and all its derivatives evaluated at 0, i.e:

$$f(x) = f(0) + f'(0) + \frac{x^2}{2}f''(0) + \dots + \frac{x^r}{r!}f^{(r)}(0) + \dots$$

let $f(x) = \arctan(\sinh(x))$ - evaluating ...using previous parts:

$$f(0) = \arctan(0)$$
$$= 0$$

$$f'(0) = \operatorname{sech}(0) = \frac{1}{\cos\dots} = \frac{1}{1} = 1$$

$$f''(0) = -(0)\tanh y$$
$$= 0$$

$f'''(0) = 1 - 2(1)^3$
$= -1$

$f^{iv}(0)$
$= 0 - 6(1)^2(0)$
$= 0$

$f^{v}(0) = (-1) - 6(1)^2(-1) - 12(1)(0)^2$
$= -1 + 6 = 5$

subbing into general Maclaurin series formula

$$\boxed{f(x) = x - \frac{x^3}{6} + \frac{x^5}{24} - \dots}$$

6. A damped spring is part of a car suspension system. In tests for the system, a mass is attached to the damped spring and is made to move upwards in a vertical line.

The motion of the system is modelled by the differential equation

$$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x=2e^{-3t}$$

where x cm is the vertical displacement of the mass above its equilibrium position and t is the time, in seconds, after motion begins.

In one particular test, the mass is moved to a position 20 cm above its equilibrium position and given an initial velocity of 1 ms^{-1} upwards. For this test, use the model to

(*a*) find an equation for x in terms of t,

(9)

(*b*) find, to the nearest mm, the maximum displacement of the mass from its equilibrium position.

(3)

In this test, the time taken for the mass to return to its equilibrium position was measured as 2.86 seconds.

© State, with justification, whether or not this supports the model.

(1)

(Total for Question 6 is 13 marks)

(a) notice we're dealing with forced harmonic motion - represented by non-homogenous 2ODE

A.E = $m^2+6m+9=0$

checking the discriminant of above quadratic to see which general solution format to use for 2ODEs

$(6)^2-4(1)(9)=36-36=0$ i.e equal roots

using $x=(A+Bt)e^{\alpha t}$

solving A.E for roots (calc equtn solver / quadratic formula)

$\alpha=\dfrac{-6\pm\sqrt{0}}{2(1)}$

$=-3$

C.F = $(A+Bt)e^{-3t}$

next choosing a P.I from table

Form of f(x)	Form of particular integral
k	λ
$ax+b$	$\lambda+\mu x$
ax^2+bx+c	$\lambda+\mu x+\nu x^2$
ke^{px}	λe^{px}
$m\cos\omega x$	$\lambda\cos\omega x+\mu\sin\omega x$
$m\sin\omega x$	$\lambda\cos\omega x+\mu\sin\omega x$
$m\cos\omega x+n\sin\omega x$	$\lambda\cos\omega x+\mu\sin\omega x$

WARNING! The particular integral must not contain any term in the complementary function. If it does, you'll need to add an x and possibly even an x^2 in front of your usual PI form initially let $x = \lambda e^{-3t}$ but looking at WARNING- can't have a multiple of C.F in the P.I $\therefore$ need

$x = \lambda t^2 e^{-3t}$

differentiate using product rule

$u = \lambda t^2 \quad v = e^{-3t}$

$u' = 2\lambda t \quad v' = -3e^{-3t}$

$$\frac{dx}{dt} = 2\lambda t(e^{-3t}) - 3e^{-3t}(3\lambda t^2)$$

.. factorise $\lambda t e^{-3t}$:

$$\frac{dx}{dt} = \lambda t e^{-3t}(2-3t)$$

differentiate above using product rule

$$\frac{d^2x}{dt^2} = 2\lambda e^{-3t} - 2(3)\lambda t e^{-3t} - 3(2)\lambda t e^{-3t} + (-3(-3)\lambda t^2 e^{-3t})$$

$$= 2\lambda e^{-3t} - 6\lambda t e^{-3t} - 6\lambda t e^{-3t} + 9\lambda t^2 e^{-3t}$$

collect like terms

$$= 2\lambda e^{-3t} - 12\lambda t e^{-3t} + 9\lambda t^2 e^{-3t}$$

factorise λe^{-3t} out

$$= \lambda e^{-3t}(2 - 12t + 9t^2)$$

subbing these derivatives into 2O.D.E

$$\lambda e^{-3t}(2-12t+9t^2) + 6((\lambda t e^{-3t})(2-3t)) + 9\lambda t^2 e^{-3t} = 2e^{-3t}$$

COMPARE COEFFICIENTS

...compare e^{-3t}:

$$2\lambda = 2$$

$\div 2 \qquad \div 2$

$$\lambda = 1$$

$\Rightarrow$ P.I $= t^2 e^{-3t}$

$\therefore$ G.S = C.F + P.I

$$= (A + Bt)e^{-3t} + t^2 e^{-3t}$$

now subbing in the initial conditions

when $t = 0$, $x = 20$

$$20 = Ae^{-3(0)} + (0)^2 e^{-3(0)}$$

$\Rightarrow 20 = A$ -①

when $t = 0$, $\frac{dx}{dt} = 100$

differentiate G.S

$$\frac{dx}{dt} = -3(A+Bt)e^{-3t} + Be^{-3t}$$
$$= -3t^2e^{-3t} + 2te^{-3t}$$

$100 = -3A + B$ -②

Sub in $A = 20$ into ②

$100 = -3(20) + B$

$\Rightarrow B = 160$

sub into previous G.S:

$x = (20 + 160t)e^{-3t} + t^2e^{-3t}$

factorise e^{-3t} out

$\boxed{= e^{-3t}(20 + 160t + t^2)}$

(b) max displacement $\Rightarrow \frac{dx}{dt} = 0$ (using product rule)

$$\frac{dx}{dt} = -3e^{-3t}(20 + 160t + t^2) + e^{-3t}(160 + 2t)$$
$$= e^{-3t}(-60 - 480t - 3t^2) + e^{-3t}(160 + 2t)$$

collect like terms

$$= e^{-3t}(100 - 478t - 3t^2) = 0$$

making each bracket equal 0

$e^{-3t} = 0$

but considering exponential properties $e^{3t} \neq 0$

$\therefore$ not a solution

$100 - 478t - 3t^2 = 0$

$\Rightarrow 3t^2 + 478t - 100 = 0$

calc equtn solver

$t = 0.208931..$

or $-159.5422..$

but $t > 0 \therefore t = 0.208931..$

subbing into x function (particular sltn)

$$x_{max} = e^{-3(0.208931..)}((20) + 160(0.208931..) + (0.2089..)^2)$$
$$= 28.57055..$$
$$\boxed{= 28.6cm \ (3 s.f)}$$

(c) subbing in $t = 2.86$,

$$x = e^{-3(2.86)}(20 + 160(2.86) + (2.86)^2) = 0.0912415..$$

at equilibrium position, $x = 0$; from our model we get a value of 0.09124..
which differs from the actual position by 0.9mm

$\boxed{\Rightarrow \text{Supports model (can be explained by inaccuracies in measurement)}}$

7.

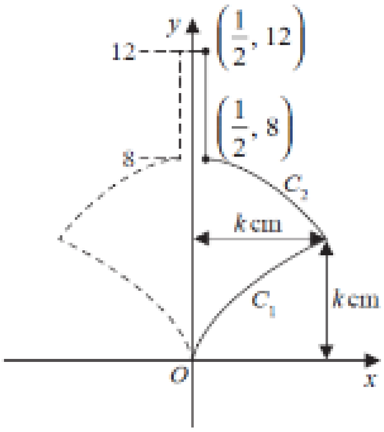

Figure 2

Figure 2 shows a sketch of the cross-section of a design for a child's spinning top. The top is formed by rotating the region bounded by the y-axis, the curve C_1, the curve C_2, the line with equation $x = \frac{1}{2}$ and the line with equation $y = 12$, through 360° about the y-axis.

The curve C_1 has equation

$$y = k^{\frac{2}{3}}x^{\frac{1}{3}} \qquad 0 \le x \le k$$

and the curve C_2 has equation

$$y = \frac{32k^3 - k - (32 - 4k)x^2}{4k^2 - 1} \qquad \frac{1}{2} \le x \le k$$

(*a*) Show that $\int_k^8 \left((4k^2-1)y - (32k^2 - k)\right) \mathrm{d}y = \frac{1}{2}(8-k)(4k^3 - 32k^2 + k - 8)$

(3)

Hence find

(*b*) the value of k that gives the maximum value for the volume of the spinning top,

(9)

(*c*) the maximum volume of the spinning top.

(3)

(Total for Question 7 is 15 marks)

(a) first part of question is just asking us to evaluate given integral

$$\int_k^8 \left((4k^2-1)y - (32k^2-k)\right)dy$$

$$\left[\frac{4k^2-1}{2}y^2 - (32k^2-k)y\right]_k^8$$

$$= \left\{\left[\frac{4k^2-1}{2}(8)^2 - (32k^2-k)(8)\right] - \left[\frac{4k^2-1}{2}(k)^2 - ((32k^2-k)k\right]\right\}$$

$$= \left\{\left[32(4k^2-1) - 8(32k^2-k)\right] - \left[\frac{k^2}{2}(4k^2-1) - k(32k^2-k)\right]\right\}$$

collect like terms

$$(32 - \frac{k^2}{2})(4k^2-1) - (8-k)(3$$

, actorise $(8-k)$ out

$$(8-k)\left[(4k^2-1)\left(4+\tfrac{1}{2}k\right)-(32k^2-k)\right]$$

expand brackets

$$(8-k)\left(16k^2+2k^3-4-\tfrac{1}{2}k-32k^2+k\right)$$

collect like terms

$$(8-k)\left(2k^3-16k^2+\tfrac{1}{2}k-4\right)$$

take ½ out

$$\boxed{\tfrac{1}{2}(8-k)(4k^3-32k^2+k-8)=RHS}$$

VOLUME =

(b) main strategy: $C_1 + C_2 +$ cylinder

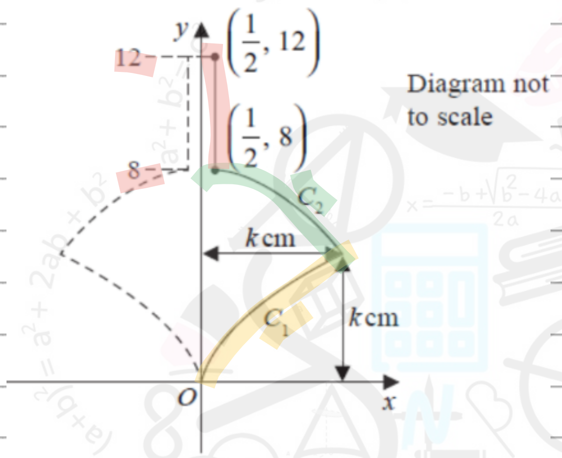

Figure 2

using formula for volume of revolution around y-axis: $V=\int_\alpha^\beta x^2\,dx$ - given

- C_1 as $x=\dfrac{y^3}{k^2}$

$$V_{C_1}=\pi\int_0^k\left(\frac{y^3}{k^2}\right)^2 dy$$

$$=\pi\int_0^k\left(\frac{y^6}{k^4}\right)dy$$

integrate

$$=\pi\left[\frac{y^7}{7k^4}\right]_0^k$$

$$=\pi\left\{\left[\frac{k^7}{7k^4}\right]-\left[\frac{0}{7k^4}\right]\right\}$$

$$=\frac{\pi k^3}{7}$$

- now for C_2:

$$y=\frac{32k^2-k-(32-4k)x^2}{4k^2-1}$$

rearrange for x^2:

$$x^2=\frac{(4k^2-1)y-3k^2+k}{-(32-4k)}$$

and subbing into formula

$$V_{C_2}:\ \frac{\pi}{4k-32}\int_k^8 (4k^2-1)y-(3k^2-k)\,dy$$

recognise rewrite integral from (a)

$$=\frac{\pi}{4k-32}\left[\tfrac{1}{2}(8-k)(4k^3-32k^2+k-8)\right]$$

$$=-\pi/8\,(4k^3-32k^2$$

CYLINDER : $\pi r^2 h$

where 'r' = $\frac{1}{2}$, 'h' = 4

$= \pi\left(\frac{1}{2}\right)^2(4) = \pi$

$\therefore$ total volume $= \frac{\pi k^3}{7} + \frac{\pi}{8}(-4k^3 - 32k^2 - k + 8) + \pi$

$\boxed{V_{max} = \frac{dv}{dk} = 0.}$

$\frac{3\pi k^2}{7} + \frac{\pi}{8}(-12k^2 + 64k - 1) = 0$

$\div \pi \times 56$

$\Rightarrow 24k^2 + 7(-12k^2 + 64k - 1) = 0$

$\Rightarrow 60k^2 - 448k + 7 = 0$

equtn solver

$k = 7.4500..$

$k = 0.0136..$

$k = 7.45$ (3 s.f)

(c) use $k = 7.45...$ in vol. form :

$V = \frac{\pi k^3}{7} + \frac{\pi}{8}(-4k^3 + 32k^2 - k - 8)$

$= \frac{\pi(7.45)^3}{7} + \frac{\pi}{8}\left(-4(7.45)^2 + 32(7.45)^2 - 7.45 - 8\right)$

$= 236.88381...$

$= \boxed{237\ cm^3\ (3\ s.f)}$